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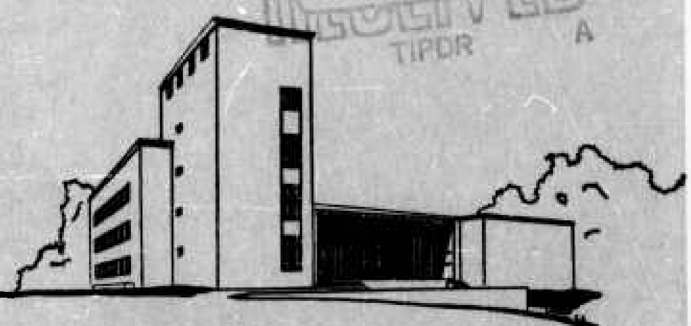

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EXTENSIONS OF A THEOREM

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NOTE

This is a revised version of a previous research report, "Extensions of a Theorem by Clark," by A. Charnes, W. W. Cooper, and G. L. Thompson, issued as ONR Research Memorandum No. 42, by the Systems Research Group, A. Charnes, Director, The Technological Institute and The Transportation Center, Northwestern University, August 10, 1961. The present report--see Section 5 ff.--extends the previous work to include the general case of mixed systems of equations and inequalities, which had not been included in the previous release.

I. Introduction

In [3], F. E. Clark established the following result:

THEOREM. If a linear programming problem [written in inequality form] has a solution, then either the primal constraint set or the dual constraint set is unbounded. The parenthetical remark is our addition to the statement of the theorem, but is implicitly assumed by Clark, as any reader of his paper will be aware. The inequality form of the general linear programming problem is:

$$
\begin{array}{lcc}
 & \text{Max } cx & \text{Min } wb \\
 & \text{subject to} & \text{subject to} \\
(1) & Ax \leq b & wA \geq c \\
 & x \geq 0 & w \geq 0
\end{array}
$$

This is sometimes referred to as the canonical form and used as a standard point of departure for further analyses.[1] Yet some care may be needed, as is true here, if a full degree of generality and understanding is to be achieved. Note, for instance, that the parenthetical remark must be included, as above, or the stated theorem is vulnerable to counter examples. For consider the following self-dual linear programming problem:

$$
\begin{array}{lcc}
 & \text{Max } 1 \cdot \xi & \text{Min } \eta \cdot 1 \\
(2.1) & \text{subject to} & \text{subject to} \\
 & \xi = 1 & \eta = 1
\end{array}
$$

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[1] Vide [4] pp. 64 ff., from which we quote, as follows: "... such problem pairs [of mixed equations and inequalities] are essentially no more general than the 'canonical' ones ..."

Here both the primal and the dual constraint set consist of exactly one point and are certainly bounded.

It is true that the latter system is not in canonical form. Also the following equivalent system, which is in canonical form,

$$\begin{array}{lcc} & \text{Max } x_1 - x_2 & \text{Min } w_1 - w_2 \\ & \text{Subject to} & \text{Subject to} \\ (2.2) & x_1 - x_2 \leq 1 & w_1 - w_2 \geq 1 \\ & -x_1 + x_2 \leq -1 & -w_1 + w_2 \geq -1 \\ & x_1, x_2 \geq 0 & w_1, w_2 \geq 0 \end{array}$$

obviously has both constraint sets non-empty and unbounded. Evidently the theorem as stated does not provide enough light to illuminate fully what is involved even for the two simple situations displayed in (2.1) and (2.2). A further development and extension therefore appears to be in order.

While it is true that every linear programming problem written in inequality form can be written as an equivalent problem in equality form, and vice versa, it is important to observe that this concept of equivalence does not imply that the constraint sets in the two ways of writing the problem are identical or even have similar properties. Examples (2.1) and (2.2) show that boundedness properties of equivalent problems may be widely different. All that the concept of equivalence implies is that from a solution of a linear programming problem in one form it is possible to obtain a solution in the other form.

In the first part of this paper we reformulate the above theorem to obtain a sharper and more general statement which we prove via a linear programming formulation that gives direct access to the duality theorem of linear programming. Then we show that the situation of (2.1) is essentially the only possible exception if the constraints involve variables that are unrestricted as to sign and involve only equality constraints. We show that any such problem is equivalent, in a sense to be defined, to a problem having these characteristics. We then return to a further variant of the above examples, (2.1) and (2.2) and then establish a general theorem for mixed systems of equations and inequalities. Finally, we consider unconstrained problems that are projection equivalent to such mixed problems.

2. Proof of the Dual Constraint Theorem

Since Clark's result is really one concerning dual sets of constraints, we shall reformulate it more generally as follows:

THEOREM 1. If $X = \{x \mid Ax \leq b,\ x \geq 0\}$ is nonempty and bounded then $W = \{w \mid wA \geq c,\ w \geq 0\}$ is nonempty and unbounded; also if W is nonempty and bounded then X is nonempty and unbounded.

PROOF. Consider the dual problems

Max ex	Min wb
subject to	subject to
$Ax \leq b$	$wA \geq e$
$x \geq 0$	$w \geq 0$

where $e = (1, 1, \ldots, 1)$. If X is nonempty and bounded then, by the dual theorem of linear programming, there exist optimal solutions x^* to the maximization problem and w^* to the maximization problem and w^* to the minimization problem. Of course, such a w^* satisfies $w^* \geq 0$ and $w^*A \geq e$.

Now let $\alpha_o = \max \{1, c_1, \ldots, c_n\}$ for any collection $c = (c_1, \ldots, c_n)$. Then for all $\alpha \geq \alpha_o$ defining $w = \alpha w^*$ we have

$$wA = \alpha\, w^*A \geq \alpha\, e \geq \alpha_o e \geq c.$$

Also, since $w^* \geq 0$ it is clear that $w \geq 0$. Thus W contains the infinite ray

$$\{w \mid w = \alpha w^*, \ \alpha \geq \alpha_o\}$$

and is thereby unbounded.

On the other hand, if $W = \{w \mid wA \geq c, w \geq 0\}$ is nonempty and bounded, we observe that equivalent expressions for X and W are:

$$W = \{w \mid (-A)^T w^T \leq -c^T, \ w^T \geq 0\}$$

$$X = \{x \mid x^T(-A)^T \geq (-b)^T, \ x^T \geq 0\},$$

where T is the transpose operation. Then, to complete the proof, we need only consider the dual linear programming problems

$$\begin{array}{ll} \text{Max} \ (-b)^T w^T & \text{Min} \ x^T(-c)^T \\ \text{Subject to} & \text{Subject to} \\ (-A)^T w^T \leq (-c)^T & x^T(-A)^T \geq (-b)^T \\ w^T \geq 0 & x^T \geq 0 \ , \end{array}$$

and apply the previous result to show that X is nonempty and unbounded.

3. Dual Problems in Unrestricted Variables

Next we consider the dual problems in unrestricted variables,

$$
\text{(3)} \qquad
\begin{array}{lcl}
\text{Max } c\xi & \qquad & \text{Min } \eta b \\
\text{subject to} & & \text{subject to} \\
A\xi = b & & \eta A = c,
\end{array}
$$

in order to prove the following theorem

THEOREM 2. If $\Xi = \{\xi \mid A\xi = b\}$ is nonempty and bounded and $H = \{\eta \mid \eta A = c\}$ is bounded, then both Ξ and H consist of a single point.

PROOF. Suppose that there could be two points, $\xi_1 \neq \xi_2$, in Ξ. Then $\xi_1 - \xi_2$ satisfies $A(\xi_1 - \xi_2) = 0$. Hence the infinite line

$$\{\xi \mid \xi = \xi_1 + \alpha(\xi_1 - \xi_2), \text{ all real } \alpha\}$$

is contained in Ξ, contradicting the boundedness which was assumed to hold for this solution set. Therefore, if Ξ is nonempty and bounded it must consist of only one point.

By the dual theorem, we immediately have H is nonempty. But then if H is also bounded, an argument which is wholly analogous to the one already given shows that H cannot consist of more than one point.

Another result concerning linear programming problems in unrestricted variables is the following.

THEOREM 3 If Ξ is nonempty and bounded then (a) Ξ and H each consist of only one point when the rows of A are linearly independent, and (b) Ξ consists of one point and H is unbounded when the rows of A are linearly dependent.

PROOF For case (a) we have, by the previous argument, that Ξ consists of only one point and H is nonempty. Suppose that H could contain two distinct points η_1, η_2. Then $\eta_1 - \eta_2 \neq 0$. But $(\eta_1 - \eta_2)A = 0$; i.e., the rows of A are linearly dependent--a contradiction. Thus H consists of only one point.

For (b) we obtain that Ξ consists of a single point and H is nonempty as before. Since the rows of A are linearly dependent, there must exist an $\tilde{\eta} \neq 0$ such that $\tilde{\eta} A = 0$. If η is any point in H then $\eta + \alpha\tilde{\eta}$ is also in H for all real α. This means H contains an infinite line and hence it is unbounded.

4. Projection Equivalence

It is very easy to change a pair of constraint sets of the type described in Theorem 2 (of which (2.1) is a specific example) into a pair of constraint sets both of which are unbounded. This can be done by merely adding a row and a column of zeros to A and adding a zero entry to each of the vectors b and c. This operation turns the constraint sets into "cylinder sets" in one higher dimension, and such sets are always unbounded.

The converse operation is also possible, namely, if A has a row or a column that is all zero and if the corresponding entry in b or c is also zero, then it is possible to project the problem into a lower dimension by eliminating the corresponding direct or dual variable.

We elaborate on this idea as follows: First recall (see [1] p. 280, Theorem 13) that if A is an $m \times n$ matrix then there exist nonsingular matrices S and T so that

$$SAT = D$$

where D is a matrix of the form

$$D = \begin{pmatrix} I_{r \times r} & 0_{r \times (n-r)} \\ 0_{(m-r) \times r} & 0_{(m-r) \times (n-r)} \end{pmatrix} \tag{4.1}$$

with I representing the identity matrix, 0 a zero matrix and the subscripts are used to indicate the dimensions in each case. Evidently the rank of A is r. Hence, if we define the projection matrices P and Q as follows

$$P_{r \times m} = (I_{r \times r} \quad 0_{r \times (m-r)})$$

$$Q_{n \times r} = \begin{pmatrix} I_{r \times r} \\ 0_{(n-r) \times r} \end{pmatrix}$$

then we see that

$$PSATQ = PDQ = I_{r \times r} \tag{4.3}$$

Here P projects m dimensional column vectors into r dimensional row vectors, while Q projects n dimensional row vectors into r dimensional row vectors.

We now return to the dual problems (3), and introduce changes in variables of the form

$$\xi = TQx \tag{5}$$

$$\eta = yPS \tag{6}$$

where P and Q are projection matrices as above (except that r here need not be the rank of A), S and T are nonsingular, and T is such that the last n-r columns of AT are all zeros and S is such that the last m-r rows of SA are all zeros. It is clear that r is greater then or equal to the rank of A.

We now want to introduce the idea of projection equivalent linear programming problems and therefore produce the following dual linear programming problems:

(7)

Max cTQx	Min yPSb
subject to	subject to
PSATQx = PSb	yPSATQ = cTQ

Then we say that the linear programming problems (3) and (7) are <u>projection equivalent</u> when we can obtain a solution to one of them from the solution to the other by means of the change of variable equations (5) and (6).

THEOREM 4 If the linear programming problem in unrestricted variables (3) has a solution then it is projection equivalent to the problem:

(8)

Max cTQx	Min yPSb
subject to	subject to
x = PSb	y = cTQ

where P, S, A, T, and Q satisfy (4.3) and r is the rank of A.

Remark. Note that the dual constraint sets of (8) each consist of a single point and hence are bounded.

PROOF. As already remarked, given A, it is possible to find P, S, T, and Q so that (4.3) is satisfied. Hence the changes of variables (5) and (6) are well defined, with r being the rank of A. The solution to (8) is obvious and the two dual problems share the common value cTQPSb. If we set

$$\xi = TQPSb \qquad \text{and} \qquad \eta = cTQPS$$

then these values will make the objective functions of (3) share a common value and this will be an optimum for (3) if these ξ and η satisfy the constraints. To show this we observe first of all that $A = S^{-1}DT^{-1}$ and that $QP = D^T$. Then we have

$$(9) \qquad A\xi = ATQPSb = S^{-1}DT^{-1}TD^TSb = S^{-1}DD^TSb = b.$$

The final reduction, indicated on the right, follows from the assumption that (3) has a solution. For, from $SA = DT^{-1}$, and the definition of D -- see (4.1) -- we observe that the last m-r rows of SA consist entirely of zeros. This implies, if (3) has a solution, that the last m-r entries of Sb consists entirely of zeros; for otherwise the equations $A\xi = b$, which are equivalent to $SA\xi = Sb$, would be inconsistent.

Thus on the assumption that (3) has a solution we have established ξ = TQPSb satisfies the constraints of this problem. Similarly, one can show that η = cTQPS satisfies the dual constraints and therefore these values for ξ and η are optimal for (3).

5. The Mixed Constraint Case

As we observed immediately after (2.2), the geometric properties for equivalent linear programming problems may be strikingly different. In fact, quite trivial alterations can bring this about. For instance, one can always replace a problem in which the rows of A are linearly independent by an equivalent problem whose matrix has linearly dependent rows--e.g., by merely repeating some of the originally stated constraints--in order to bring Theorem 3 into play and alter the dual solution set from one which is bounded to one which is unbounded.

Preparatory to dealing with the general case of mixed equations and inequalities, we exhibit still another variation of (2.1) and (2.2). Evidently the problem

$$\text{Max } \xi$$

Subject to

(10.1) $$\xi = 1$$
$$\xi \geq 0$$

does not alter any solution property for the direct problem in (2.1). The only point in Ξ continues to be $\xi = 1$. The dual to (10.1), viz.,

$$\text{Min } \eta$$

(10.2) Subject to

$$\eta \geq 1$$

admits of an unbounded solution set, H, so that the mere adjunction of a redundant non-negativity requirement produces a markedly different situation from the dual to (2.1).

It is this kind of situation that Clark's theorem fails to illuminate and so we now elaborate for the general situation as follows.

THEOREM 5 If Ξ is nonempty and bounded for

$$\text{Max} \quad c_1 \xi_1 + c_2 \xi_2$$

Subject to

$$A_{11} \xi_1 + A_{12} \xi_2 = b_1$$

$$A_{21} \xi_1 + A_{22} \xi_2 \leq b_2$$

$$\xi_2 \geq 0$$

then H is unbounded for

$$\text{Min } \eta_1 b_1 + \eta_2 b_2$$

Subject to

$$\eta_1 A_{11} + \eta_2 A_{21} = c_1$$

$$\eta_1 A_{12} + \eta_2 A_{22} \geq c_2$$

$$\eta_2 \geq 0.$$

PROOF. If Ξ is nonempty and bounded then a finite optimum exists for every μ in the problem

$$\text{Max } (c_1 \xi_1 + \mu e_2 \xi_2)$$

Subject to

$$A_{11} \xi_1 + A_{12} \xi_2 = b_1$$

$$A_{21} \xi_1 + A_{22} \xi_2 \leq b_2$$

$$\xi_2 \geq 0 \ ,$$

where μ is a scalar, A_{ij}, for $i, j = 1, 2$, is a partitioning of the matrix A, and e_2 is, as before, a vector with all components equal to unity. By the duality theorem, the following problem then also has a solution for every μ:

$$\text{Min } \eta_1 b_1 + \eta_2 b_2$$

Subject to

$$\eta_1 A_{11} + \eta_2 A_{21} = c_1$$

$$\eta_1 A_{12} + \eta_2 A_{22} \geq \mu e_2$$

$$\eta_2 \geq 0$$

Since $K e_2 \geq c_2$ for some $K > 0$, every solution with $\mu \geq K$ is, a fortiori, a solution of

$$\eta_1 A_{11} + \eta_2 A_{21} = c_1$$

$$\eta_1 A_{12} + \eta_2 A_{22} \geq c_2$$

$$\eta_2 \geq 0,$$

and hence is in H.

Suppose, contrary to the assertion of the theorem, that H were bounded. But H bounded implies there exist

$K_i > 0$, $i = 1, 2$, such that $-K_i e_1 \leq \eta_i \leq K_i e_2$ for all $(\eta_1, \eta_2) \in H$. And also

$$\eta_1 A_{12} + \eta_2 A_{22} \leq Me_2$$

for some $M > 0$. This, however, contradicts the assumed existence of a solution satisfying

$$\eta_1 A_{12} + \eta_2 A_{22} \geq 2\,Me_2$$

Hence H must be unbounded. Q. E. D.

Our final result shows that it is possible to reformulate any linear programming problem that has a solution in a finite number of ways so that both its constraint sets are bounded one-point sets.

THEOREM 6. Any solvable linear programming problem is projection equivalent to a finite number of linear programming problems having one-point (hence bounded) dual constraint sets.

PROOF. Suppose that the dual constraints of the problem are $A\xi \leq b$ and $\eta A \geq c$, where, as in the statement of Theorem 5, some of the constraints are equalities and some inequalities, and the nonnegativity conditions are correspondingly determined. Let ξ^o and η^o be solutions to the linear programming problem. Then these two vectors satisfy the above constraints, together with any nonnegativity constraints, and also satisfy $\eta^o b = c\xi^o$. But then

$$\eta^o A \xi^o \leq \eta^o b = c\xi^o \leq \eta^o A \xi^o$$

This means that whenever one of the constraints is satisfied as a strict inequality the corresponding dual variable is necessarily zero. Now let A', b', and c' be the corresponding matrices with rows or columns deleted corresponding to zero components of ξ^o and η^o. Let $\xi^{o'}$ and $\eta^{o'}$ be similarly defined. Then by the above remarks we have

$$A'\xi^{o'} = b', \quad \eta^{o'}A' = c', \text{ and } \eta^{o'}b' = c'\xi^{o'}.$$

and also any remaining nonnegativity conditions are still satisfied. Hence $\xi^{o'}$ and $\eta^{o'}$ are solutions to the unconstrained dual problems

$$\begin{array}{ll} \text{Max } c'\xi' & \text{, Min } \eta' b' \\ \text{Subject to} & \text{Subject to} \\ A'\xi' = b' & \eta' A' = c'. \end{array}$$

Moreover, among the solutions to the latter unconstrained problem will be found ones which satisfy the desired nonnegativity conditions that may remain, and hence will be solutions to the original problem. Thus the unconstrained problem is projection equivalent to the original problem, in the sense that among the solutions to each problem appear solutions to the other.

We can now apply Theorem 4 and make further projection equivalence of the unconstrained problem just found with one whose dual constraint sets are one-point.

Finally, we remark that although there can be infinitely many solutions to a linear programming problem it is evident that there are only a finite number of ways of crossing out

rows and columns, hence the above described process will lead to only a finite number of different projection equivalent problems.

6. Conclusion

Via (3) and (8) and their associated theorems, we have attempted to illuminate one aspect of the geometric properties associated with dual pairs of linear programming problems. These results, together with Theorems 5 and 6, make it possible either to remove or to adjoin various constraining conditions in order to obtain suitable geometric properties for the sets Ξ and H. In many cases, it is desirable to secure proofs of various theorems by constructing so-called linear programming chains[1/] which alter the geometric properties of direct or dual solution sets. The above theorems can be used for guidance in securing the geometric properties that are wanted in these chains of "equivalent" problems.

There are obvious connections between the "extended theorem of the alternative" (see [2], pp. 250 ff. and p.441) and the first projection employed in the proof of Theorem 6. This, in turn, means that a variety of interesting applications (e.g., in economics) can be made from this and the related developments in this paper. We cannot pursue this topic here, however. Instead, we conclude by observing that no real trouble emerges from these developments as far as computations or solutions are concerned. Every constructive method for solving linear programming

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1/ See [2].

problems is now designed so that it does not require one to examine more than a bounded set. Indeed, because of so-called regularization procedures[1/] that have now been developed, the computation may be prosecuted in a routine fashion to obtain all relevant information even when a problem is not solvable or has an infinite optimum, fails to have full rank, and so on.

1/ Vide, e.g., Chapter VI and VII in [2].

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